A NON-CONTINUOUS "STEINER POINT"

BY

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ABSTRACT

A new function is constructed on the space of compact, convex sets which has all the standard properties of the Steiner point except for continuity.

1. Introduction.

Let κ^d denote the class of compact, convex subsets of E^d, d-dimensional Euclidean space. With each $K \in \kappa^d$, we may associate a point, the *Steiner point*, $s(K)$ by:

$$s(K) = \frac{1}{\omega_d} \int h(K,u)u \, du$$

where $h(K,u)$ is the support function of K in direction u, ω_d is the volume of the unit n-ball, and the integral is taken over S^{d-1}, the unit $(d-1)$-sphere. Since this definition was introduced by Shephard [11], various properties (see [3], p. 314, [7] and [12]) of $s(K)$ have been discovered:

(1) s is continuous with respect to ρ, the Hausdorff metric;

(2) $s(K + L) = s(K) + s(L)$, where addition on the left is Minkowski addition;

(3) For any similarity transformation $\sigma, \sigma(s(K)) = s(\sigma(K))$;

(4) $s(K \cap L) + s(K \cup L) = s(K) + s(L)$ if $K, L, K \cup L \in \kappa^d$;

(5) $\Sigma_{i=0}^{d}(-1)^i \Sigma s(F_j^i) = s(P)$ for any d-polytope P where the inner summation on the left is taken over all i-dimensional faces of P.

In [2], the question was raised, if $f: \kappa^d \to E^d$ is a function satisfying (2) and (3), is $f(K) = s(K)$? More generally, which combinations of the above properties will characterize the Steiner point? Many authors have attacked various aspects of this problem ([1], [4], [5], [6], [9], [12]) culminating in Schneider's proof [10]

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that properties (1), (2) and (3) will characterize the Steiner point. Here we show that all three conditions are necessary by exhibiting a function which satisfies properties (2), (3) and (4), but is not the Steiner point.

In case the definition of the next section appears too unmotivated, it was suggested by the following function on a 3-polytope P:

$$f(P) = \Sigma\, l(E_i)\theta(E_i)\, u(E_i)$$

where $l(E_i)$ is the length of the edge E_i, $\theta(E_i)$ is its exterior angle, $u(E_i)$ is the unit vector orthogonal to E_i pointing in such a direction as to bisect the exterior angle, and the sum is taken over all edges of P.

2. **Construction.**

For any $K \in \kappa^d$ and any direction $v \in S^{d-1}$, let $F(K,v)$ denote the *face* of K with outer normal, v that is, $F(K,v) = K \cap H(K,v)$ where $H(K,v)$ is the supporting hyperplane to K in direction v. Let $\delta(K,v) =$ diameter of $F(K,v)$. Let $E(K,v)$ be the set of all edges of K in direction v which do not lie in any j-dimensional face [*j-face*] of K for $j \geqq 2$ and let $S(K,v)$ be the set of support directions of $E(K,v)$. It is easy to show that $S(K,v)$ is a (Hausdorff)-measurable subset of the $(d-2)$-sphere on S^{d-1} which is orthogonal to v. Finally, let ω be the usual $(d-2)$ surface area measure.

Consider the following functions:

$$u(K,v) = \int_{S(K,v)} \delta(K,w)w\, d\omega \tag{6}$$

and

$$f(K) = \sum_{v \in S^{d-1}} u(K,v). \tag{7}$$

We claim that the function $s'(K) = s(K) + f(K)$ is additive and commutes with similarity transformations—that is, statisfies properties (2) and (3), but is not continuous. There are several items which must be checked and we will leave the most fundamental question—the finiteness of f until last.

Assuming f is always finite, however, it is clear that s' is not continuous. For if $R \in \kappa^d$ is *rotund* (that is, has no line segments in its boundary), then $f(R) = 0$. But since there exist K with $f(K) \neq 0$—for example, half-discs—and since the rotund bodies are dense in κ^2, s' is not continuous.

To prove that s' commutes with similarity transformations we observe first that $s'(\lambda K) = \lambda s'(K)$ for any $\lambda > 0$, so we may restrict our attention to rigid motions. Now every rigid motion in E^d can be generated by translations and by reflections across hyperplanes which contain the origin. So suppose τ is any translation, say $\tau K = K + x_0$. Then $s(\tau K) = s(K) + x_0$. But $f(\tau K) = f(K)$ since $\delta(\tau K, v) = \delta(K, v)$ for all v. Hence

$$s'(\tau K) = s(K) + x_0 + f(K) = s'(K) + x_0 = \tau s'(K).$$

If π is any hyperplane through the origin and ρ denotes reflection across π, then $\rho^2 = 1$. Moreover, $\delta(\rho K, \rho w) = \delta(K, w)$ and hence, $\delta(\rho K, w) = \delta(K, \rho w)$.

Thus

$$\begin{aligned} u(\rho K, \rho v) &= \int_{S(\rho K, \rho v)} \delta(\rho K, w)\, w\, d\omega = \int_{S(\rho K, \rho v)} \delta(K, \rho w) \rho(\rho w)\, d\omega \\ &= \rho \int_{S(\rho K, \rho v)} \delta(K, \rho w) \rho w\, d\omega = \rho \int_{S(K, v)} \delta(K, w') w'\, d\omega = \rho u(K, v). \end{aligned}$$

Hence

$$f(\rho K) = \sum_{\rho v \in S^{d-1}} u(\rho K, \rho v) = \rho \sum_{\rho v \in S^{d-1}} u(K, v) = \rho f(K).$$

Now suppose that K has been translated so that $s(K) = 0$. Then $s(\rho K) = 0$ and $s'(\rho K) = f(\rho K) = \rho f(K) = \rho s'(K)$, so s' commutes with ρ. By our earlier remarks, this proves that s' commutes with every similarity transformation.

Our next task, establishing the additivity of s', is somewhat easier. Since s itself is additive, it clearly suffices to establish the additivity of f. In fact, it suffices to show that $u(K + L, v) = u(K, v) + u(L, v)$ if $K, L \in \kappa$. Recall that $F(K + L, w) = F(K, w) + F(L, w)$ and in particular, if $F(K + L, w)$ is an edge of $K + L$ in direction v, then $F(K, w)$ and $F(L, w)$ are parallel edges (in direction v) or else one is a point. In either case, $\delta(K + L, w) = \delta(K, w) + \delta(L, w)$. Moreover, the above argument implies that $S(K + L, v) = S(K, v) \cup S(L, v)$. Note that we may also compute $u(K, v)$ and $u(L, v)$ by integrating over $S(K + L, v)$ since any additional directions correspond to vertices and thus do not contribute to the integral.

Hence

$$\begin{aligned} u(K + L, v) &= \int_{S(K+L, v)} \delta(K + L, w) w\, d\omega = \int_{S(K+L, v)} [\delta(K, w) + \delta(L, w)] w\, d\omega \\ &= u(K, v) + u(L, v). \end{aligned}$$

All that remains is showing that $|f(K)| < \infty$ for all $K \in \kappa^d$. For this, it is sufficient to show that $\Sigma|u(K,v)| < \infty$ where the summation is over S^{d-1}. Since $|u(K,v)| \leqq u'(K,v) = \int_{S(K,v)}|\delta(K,w)|\, d\omega$ it suffices to establish a bound on $\Sigma u'(K,v)$.

To do this, some additional notation will be useful. Let π_v denote projection on the subspace orthogonal to v, and let B be the d-ball of radius 1. Finally, let $\alpha(v)$ denote the $(d-1)$-dimensional surface area of $E(K^*,v)$ where $K^* = K + B$ {recall that $E(K^*,v)$ is the set of edges on K^* in direction v which do not lie in any j-face of K^* for $j \geqq 2$}. The idea of the proof is to establish that $\alpha(v) \geqq u'(K^*,v)$, whence it will follow that $\frac{1}{2}\Sigma u'(K,v) \leqq (d-1)$-dimensional surface area of K^*, and thus is finite.

We begin by supposing that each member of $E(K^*,v)$ has length λ. Then it is clear that $\alpha(v) = \lambda\omega(\pi E(K^*,v))$ and that $u'(K,v) = \lambda\omega(\pi S(K^*,v))$ {$S(K^*,v)$ is the set of support directions of $E(K^*,v)$}. But if we consider $S(K^*,v)$ as a subset of the boundary of B, then $\omega(\pi E(K^*,v)) \geqq \omega(\pi E(K,v)) + \omega(\pi S(K^*,v))$. This last inequality is well-known for the total surface area of a sum of two convex bodies and the usual proofs carry over when we restrict our attention to a fixed set of support directions. Since $S(K,v) = S(K^*,v)$, the result is proved for the special case.

In the general case, we may write

$$\alpha(v) = \int_{\pi E(K^*,v)} \delta(\pi^{-1}x)d\omega(x), \qquad u'(K,v) = \int_{\pi S(K^*v)} \delta(\pi^{-1}x)d\omega(x),$$

where δ denotes length of the pre-image. Then we break up $E(K^*,v)$ into subsets having length between i/n and $(i+1)/n$ and let $n \to \infty$. A standard approximation argument applies to complete the proof that $\alpha(v) \geqq u'(K,v)$.

Now a subset A of K^* belongs to $E(K^*,v)$ and $E(K^*,v')$, where $-v \neq v'$, only if A lies on a j-face of K^* for $j \geqq 2$. But this is impossible by our definition of $E(K^*,v)$. Hence $\Sigma u'(K,v) \leqq \Sigma\alpha(v) \leqq 2$ area $(K^*) < \infty$.

This completes the proof.

3. **Remarks.**

Given two "Steiner points" s and s', an entire family may be generated by the formula

$$s^*(K) = \alpha s(K) + (1-\alpha)s'(K).$$

It is easily checked that s^* satisfies both (2) and (3) if s and s' do.

I am grateful to G. D. Chakerian for informing me that the function f is not new, at least for 3-dimensional polytopes. It appears in [14, p. 165] and is shown to vanish. Slight modifications of the proof given there serve to establish that if P is any d-dimensional polytope $f(P) = 0$. Thus $s(K) = s'(K)$ not only if K is a rotund body, but also if K is a polytope.

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